

VISCOSITY SOLUTION OF HAMILTON-JACOBI EQUATION BY A LIMITING MINIMAX METHOD

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ABSTRACT. For non convex Hamiltonians, the viscosity solution and the more geometric minimax solution of the Hamilton-Jacobi equation do not coincide in general. They are nevertheless related: we show that iterating the minimax procedure during shorter and shorter time intervals one recovers the viscosity solution.

1. INTRODUCTION

Consider the Cauchy problem for the Hamilton-Jacobi equation

$$(H-J) \quad \begin{cases} \partial_t u(t, x) + H(t, x, \partial_x u) = 0, \\ u(0, x) = v(x) \end{cases}$$

There are in general no global classical C^1 solutions of the Cauchy problem (H-J), due to the crossing of characteristics. The only solutions that exist are “weak solutions” in the sense of distributions, for example functions verifying the equation almost everywhere. However, such solutions are not unique. Different attempts have been made, adding conditions on weak solutions to ensure uniqueness and, of course, some physical meaning. Roughly, there are two directions in which the pioneers worked: for conservation laws, there are entropy conditions, such as Oleinik’s in dimension one [Lax73], and Kruzkov’s for general dimensions [Kru70]; for equations with convex Hamiltonian (or initial functions), there are the explicit solution constructed by Hopf formula [Hop65] for conservation laws and the Lax-Oleinik formula for general Hamiltonians, which are widely used in weak KAM theory, see for example [Fat].

In the 1980’s, M. G. Crandall, L. C. Evans, and P. L. Lions introduced the notion of “viscosity solution” for general nonlinear first order partial differential equations [Lio82, CEL84]. Viscosity solutions need not be differentiable anywhere, which makes their relationship with the classical crossing of characteristics unclear. However, they possess very general existence, uniqueness and stability properties and, in a large class of “good” cases, they coincide with the weak solutions introduced before them.

For general Hamiltonians, with the development of symplectic geometry, M. Chaparon [Cha91] has extended the point of view of Lax and proposed a geometric method to construct an explicit weak solution by taking the minmax of a generating family of the geometric solution. Then follows the works of T. Joukovskaia, A. Ottolenghi, C. Viterbo, F. Cardin, [Jou93, Vit96, Vit06, VO95, BC11]. It is shown that this minmax solution coincide with the viscosity solution for convex Hamiltonians [Jou93]. However, there are counterexamples showing that they may differ for non-convex

Hamiltonians. Indeed, by the uniqueness of viscosity solutions, they must possess a semi-group property with respect to time, which is not a general feature of the minmax.

In this paper, we provide a new formula for the construction of weak solutions for the (H-J) equation, called iterated minmax, which is obtained by dividing a given time interval into small pieces and taking the minmax step by step. Our main purpose is to show that as the small time intervals go to zero, one gets a limit solution which is indeed the viscosity solution (see Theorem 3.23). This answers a question of Marc Chaperon.

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2. GENERATING FAMILIES AND MINMAX SELECTOR

We will first give a brief survey of the classical theory, for a closed manifold M , of generating families for Lagrangian submanifolds $L \subset T^*M$ and then introduce the minmax selector, which serves to extract a section from the Lagrangian. Next we pass to the model case $M = \mathbb{R}^d$ where we will generalize the classical notions, on the one hand, to fit the non-compactness of manifolds, and on the other hand, to the Lipschitz case, where we do not have smooth Lagrangian submanifolds but the notions of generating family and minmax still exist.

2.1. General theory for closed manifold.

Definition 2.1. A *generating family* for a Lagrangian submanifold $L \subset T^*M$ is a C^2 function $S : M \times \mathbb{R}^k \rightarrow \mathbb{R}$ such that 0 is a regular value of the map $(x, \eta) \mapsto \partial S(x, \eta)/\partial \eta$ and

$$L = \left\{ \left(x, \frac{\partial S}{\partial x}(x, \eta) \right) : \frac{\partial S}{\partial \eta} = 0 \right\};$$

more precisely, the condition that 0 is a regular value implies that the *critical locus* $\Sigma_S := \{(x, \eta) | \partial_\eta S = 0\}$ is a submanifold and that the map

$$i_S : \Sigma_S \rightarrow T^*M, \quad (x, \eta) \mapsto (x, \partial_x S(x, \eta))$$

is an immersion; we require that i_S be an embedding and, of course, $i_S(\Sigma_S) = L$.

A function S on $M \times \mathbb{R}^k$ need not have critical points. However, it does have critical points if we prescribe some behavior at infinity as in the following definition:

Definition 2.2. A generating family $S : M \times \mathbb{R}^k \rightarrow \mathbb{R}$ is *quadratic at infinity* if

$$S(x, \eta) = \psi(x, \eta) + Q(\eta)$$

where Q is a nondegenerate quadratic form and $S = Q$ outside a compact set.

The existence of a GFQI is invariant under Hamiltonian isotopy.

Theorem 2.3 (Sikorav [Sik87]). *Suppose L_0 and L_1 are two Lagrangian submanifolds of T^*M which are Hamiltonianly isotopic, and L_0 admits a GFQI, then so does L_1 . In particular, any Lagrangian manifold Hamiltonianly isotopic to the zero section 0_{T^*M} admits a GFQI.*

Note that the generating families are not unique. Let $S : M \times \mathbb{R}^k \rightarrow \mathbb{R}$ be a generating family of L , then one can obtain another family \tilde{S} generating the same L by

- (a) Fiberwise diffeomorphism : $\tilde{S}(x, \eta) := S(x, \varphi(x, \eta))$, where $(x, \eta) \mapsto (x, \varphi(x, \eta))$ is a fiberwise diffeomorphism.
- (b) Adding a constant: $\tilde{S}(x, \eta) := S(x, \eta) + C$.
- (c) Stabilization: $\tilde{S}(x, \eta, \xi) := S(x, \eta) + q(\xi)$, where q is a nondegenerate quadratic form.

Theorem 2.4 (Viterbo, Théret [Thé99]). *If a Lagrangian submanifold $L \subset T^*M$ is Hamiltonianly isotopic to the zero section 0_{T^*M} , then L admits a unique GFQI up to the above operations.*

Now given a Lagrangian submanifold $L \subset T^*M$ with a GFQI

$$S : M \times \mathbb{R}^k \rightarrow \mathbb{R}, \quad S(x, \eta) = \psi(x, \eta) + Q(\eta)$$

consider the sub-level sets

$$S_x^a := \{\eta : S(x, \eta) \leq a\},$$

the homotopy type of (S_x^a, S_x^{-a}) does not depend on a when a is large enough, we may write it as $(S_x^\infty, S_x^{-\infty})$. If the Morse index of Q is k_∞ , then

$$H_i(S_x^\infty, S_x^{-\infty}; \mathbb{Z}_2) = H_i(Q^\infty, Q^{-\infty}; \mathbb{Z}_2) \simeq \begin{cases} \mathbb{Z}_2, & i = k_\infty \\ 0, & \text{otherwise} \end{cases}$$

Definition 2.5. The *minmax* function is defined as

$$R_S(x) := \inf_{[\sigma]=A} \max_{\eta \in \sigma} S(x, \eta)$$

where A is a generator of the homology group $H_{k_\infty}(S_x^\infty, S_x^{-\infty}; \mathbb{Z}_2)$. A relative cycle σ of class A is called a descending cycle.

We can also introduce the maxmin function by considering the homology group defined by upper level sets:

$$H_{k'_\infty}(X \setminus S_x^{-\infty}, X \setminus S_x^\infty; \mathbb{Z}_2) \simeq \mathbb{Z}_2$$

where $k'_\infty = k - k_\infty$ and $X = \mathbb{R}^k$ is the fiber space.

Definition 2.6. The *maxmin* function is defined as

$$P_S(x) := \sup_{[\sigma]=B} \min_{\eta \in \sigma} S(x, \eta)$$

where B is a generator of the homology group $H_{k'_\infty}(X \setminus S_x^{-\infty}, X \setminus S_x^\infty; \mathbb{Z}_2)$. A relative cycle σ of class B is called an ascending cycle.

Remark 2.7. The minmax et maxmin are defined fiberwise for generating families. We remark that they are well-defined for functions $f : X \rightarrow \mathbb{R}$ “quadratic at infinity” in the sense that the critical set of f is compact and $(f^\infty, f^{-\infty})$ has the homotopy type of $(Q^\infty, Q^{-\infty})$ for a nondegenerate quadratic form Q . For example, the condition is satisfied if the derivative of $f - Q$ is bounded. This is a simple generalization of functions exactly quadratic at infinity which requires $f = Q$ outside a compact set.

By the uniqueness theorem 2.4, for a given Lagrangian submanifold L , the minmax and maxmin are independent of the GFQI, up to a constant.

Proposition 2.8 ([WEI11]). *The minmax and the maxmin are equal, i.e. $R_S(x) = P_S(x)$.*

The minmax defines almost everywhere a section of the projection $T^*M \rightarrow M$ restricted to L (“graph selector”):

Theorem 2.9 (Sikorav, Chaperon [Cha91, PPS03]). *Suppose $L \subset T^*M$ admits a generating family quadratic at infinity S , then R_S is a Lipschitz function and there exists an open set $\Omega \subset M$ with full measure such that for $x \in \Omega$,*

$$(x, dR_S(x)) \in L.$$

2.2. The case $M = \mathbb{R}^d$. In the rest of the paper, we will take the manifold M to be \mathbb{R}^d , in which case the generating families are constructed explicitly. For a general manifold, one can embed it into some \mathbb{R}^d and use the trick of Chekanov [Che96, Bru91] to obtain generating families from those in \mathbb{R}^d .

2.2.1. Construction of generating functions and phases. In the following, we equip \mathbb{R}^k with the Euclidean ℓ^2 norm $|\cdot|$, and matrices in \mathbb{R}^k with the associated operator norm. We denote by $\text{Lip}(f)$ the Lipschitz constant of a function f and by $\pi : T^*\mathbb{R}^d \rightarrow \mathbb{R}^d$ the canonical projection $\pi(x, y) = x$.

We denote by $H : [0, T] \times T^*\mathbb{R}^d \rightarrow \mathbb{R}$ a C^2 Hamiltonian satisfying

$$(2.1) \quad c_H := \sup |D^2 H_t(x, y)| < \infty$$

and by X_{H_t} the associated time-dependent Hamiltonian vector field. By the general theory of differential equations, as $c_H = \max_t \text{Lip}(DH_t) = \max_t \text{Lip}(X_{H_t})$, the Hamiltonian transformation $\varphi_H^{s,t}$ obtained by integrating X_{H_τ} from $\tau = s$ to $\tau = t$ is a well-defined diffeomorphism for all $(s, t) \in [0, T]$. For simplicity, we sometimes write $\varphi_s^t = (X_s^t, Y_s^t) := \varphi_H^{s,t}$ without mentioning H .

We will be mostly interested in the special case where H has compact support, and consider the Lagrangian submanifolds of $T^*\mathbb{R}^d$ which are Hamiltonianly isotopic to the zero section:

$$\mathcal{L} := \{L = \varphi(dv), \quad v \in C^2 \cap C^{\text{Lip}}(\mathbb{R}^d), \varphi \in \text{Ham}_c(T^*\mathbb{R}^d)\};$$

here $C^{\text{Lip}}(\mathbb{R}^d)$ denotes the space of globally Lipschitz functions and

$$\begin{aligned} dv &:= \{(x, dv(x)), x \in \mathbb{R}^d\} \subset T^*\mathbb{R}^d \\ \text{Ham}_c(T^*\mathbb{R}^d) &= \{\varphi = \varphi_H, \quad H \in C_c^2([0, 1] \times T^*\mathbb{R}^d)\} \end{aligned}$$

where $\varphi_H = \varphi_H^{0,1}$ is the endpoint of the isotopy (“Hamiltonian flow”) defined by H .

Definition 2.10. A diffeomorphism $\varphi : T^*\mathbb{R}^d \rightarrow T^*\mathbb{R}^d$ admits a *generating function* ϕ , if $\phi : T^*\mathbb{R}^d \rightarrow \mathbb{R}$ is of class C^2 , such that $((x, y), (X, Y)) \in \text{Graph}(\varphi)$ if and only if

$$\begin{cases} x = X + \partial_y \phi(X, y) \\ Y = y + \partial_X \phi(X, y). \end{cases}$$

This can be interpreted as follows: the isomorphism

$$\begin{aligned} I : T^*\mathbb{R}^d \times T^*\mathbb{R}^d &\rightarrow T^*(T^*\mathbb{R}^d) \\ (x, y, X, Y) &\mapsto (X, y, Y - y, x - X) \end{aligned}$$

is symplectic if $T^*\mathbb{R}^d$ is equipped with the standard symplectic form $\omega = dx \wedge dy$ and $T^*\mathbb{R}^d \times T^*\mathbb{R}^d$ with the symplectic form $(-\omega) \oplus \omega = dX \wedge dY - dx \wedge dy$; this symplectic isomorphism I sends the diagonal of the space $T^*\mathbb{R}^d \times T^*\mathbb{R}^d$ to the zero section of the cotangent space $T^*(T^*\mathbb{R}^d)$ and $\text{Graph}(\varphi)$ to $\text{Graph}(d\phi)$.

Hence, if it exists, the generating function ϕ is unique up to the addition of a constant.

Lemma 2.11 ([Cha90]). *If*

$$\delta_H := c_H^{-1} \ln 2$$

then, for $|s - t| < \delta_H$, the map

$$\alpha_s^t : (x, y) \mapsto (X_s^t(x, y), y)$$

is a diffeomorphism. As a consequence, φ_s^t admits the generating function

$$\phi_s^t(X_s^t, y) = \int_s^t \left((Y_s^\tau - y) \dot{X}_s^\tau - H(\tau, X_s^\tau, Y_s^\tau) \right) d\tau$$

Proposition 2.12 (Composition formula [Sik87]). *If a Lagrangian submanifold $L_0 \subset T^*\mathbb{R}^d$ admits a generating family $S_0 : \mathbb{R}^d \times \mathbb{R}^k \rightarrow \mathbb{R}$, then for $|t - s| < \delta_H$, the Lagrangian submanifold $\varphi_s^t(L_0)$ has the generating family*

$$(2.2) \quad S(x, (\xi, x_0, y_0)) = S_0(x_0, \xi) + \phi_s^t(x, y_0) + xy_0 - x_0y_0$$

The proof is straightforward.

Corollary 2.13. *For each subdivision $0 \leq s = t_0 < t_1 \cdots < t_N = t \leq T$ satisfying $|t_i - t_{i+1}| < \delta_H$, if $\phi_H^{t_i, t_{i+1}}$ is the generating function of $\varphi_H^{t_i, t_{i+1}}$ defined in Lemma 2.11, we have the following for each C^2 function $v : \mathbb{R}^d \rightarrow \mathbb{R}$:*

- i) *A generating family $S : \mathbb{R}^d \times (T^*\mathbb{R}^d)^N \rightarrow \mathbb{R}$ of the Lagrangian submanifold $\varphi_H^{s, t}(dv)$ is*

$$(2.3) \quad S(x, \eta) = v(x_0) + \sum_{0 \leq i < N} \phi_H^{t_i, t_{i+1}}(x_{i+1}, y_i) + \sum_{0 \leq i < N} (x_{i+1} - x_i)y_i,$$

where $x_N := x$, $\eta = ((x_i, y_i))_{0 \leq i < N}$.

- ii) *One defines a C^2 family $S : [s, t] \times \mathbb{R}^d \times (T^*\mathbb{R}^d)^N \rightarrow \mathbb{R}$ such that each $S_\tau := S(\tau, \cdot)$ is a generating family for $\varphi_H^{s, \tau}(dv)$ as follows: for each $\tau \in [t_j, t_{j+1})$,*

$$(2.4) \quad S(\tau, x, \eta) := v(x_0) + \phi_H^{t_j, \tau}(x_{j+1}, y_j) + \sum_{0 \leq i < j} \phi_H^{t_i, t_{i+1}}(x_{i+1}, y_i) + \sum_{0 \leq i < N} (x_{i+1} - x_i)y_i.$$

iii) For each critical point η of $S(\tau, x; \cdot)$, the corresponding critical value is

$$S_\tau(x; \eta) = v(x_0) + \int_s^\tau \left(Y_s^\sigma \dot{X}_s^\sigma - H(\sigma, X_s^\sigma, Y_s^\sigma) \right) d\sigma,$$

where $X_s^\sigma := X_s^\sigma(x_0, dv(x_0))$ and $Y_s^\sigma := Y_s^\sigma(x_0, dv(x_0))$. Hence, the critical values of $S(\tau, x; \cdot)$ are the real numbers

$$(2.5) \quad v(X_\tau^s(z)) + \int_s^\tau \left(Y_\tau^\sigma(z) \dot{X}_\tau^\sigma(z) - H(\sigma, X_\tau^\sigma(z), Y_\tau^\sigma(z)) \right) d\sigma$$

with $z := (x, y)$, $y \in \pi^{-1}(x) \cap \varphi_H^{s, \tau}(dv)$.

Proof. i) As the Hamiltonian flow is a “two-parameter groupoid”, we have that

$$\varphi_H^{s, t} = \varphi_H^{t_0, t_N} = \varphi_H^{t_{N-1}, t_N} \circ \dots \circ \varphi_H^{t_0, t_1};$$

hence, if $|t_{i+1} - t_i| < \delta_H$ for all i , it follows from the composition formula in Proposition 2.12 that formula (2.3) does define a generating family for $\varphi_H^{s, t}(dv)$.

ii) is clear.

iii) is proved by inspection (and very important). \square

2.2.2. Generating families quadratic at infinity. We now give a weaker definition of “quadratic at infinity”, which will include such cases and take into account the non compactness of the base manifold.

Definition 2.14. A family $S : M \times \mathbb{R}^k \rightarrow \mathbb{R}$ is called *quadratic at infinity* if there exists a nondegenerate quadratic form $Q : \mathbb{R}^k \rightarrow \mathbb{R}$ such that, for any compact subset $K \subset M$, the restriction $S|_{K \times \mathbb{R}^k}$, modulo a fiberwise diffeomorphism, equals Q off a compact set.

Proposition 2.15 (Proposition 1.20, [WEI13]). *Suppose a family $S : \mathbb{R}^d \times \mathbb{R}^k \rightarrow \mathbb{R}$ is of the form*

$$S(x, \eta) = \psi(x, \eta) + Q(\eta) := \ell(x, \eta) + \psi_1(x, \eta) + Q(\eta)$$

where $Q(\eta) = \frac{1}{2}\eta^T B \eta$ is a nondegenerate quadratic form, ℓ is a C^2 function such that $\partial_\eta \ell$ is bounded in $K \times \mathbb{R}^k$ for each compact $K \subset \mathbb{R}^d$, and ψ_1 is C^2 with

$$(2.6) \quad c := \sup |\partial_\eta^2 \psi_1(x, \eta)| < |B^{-1}|^{-1}.$$

Then S is quadratic at infinity in the new sense.

Note that a necessary condition for L to admit a GFQI in our new sense is that, for any K , the intersection $L \cap \pi^{-1}(K)$ be compact and *nonempty*: indeed, a function on \mathbb{R}^k equal to a nondegenerate quadratic form off a compact set must have critical points.

Corollary 2.16. *If H has compact support, the generating phases constructed in Corollary 2.13 are quadratic at infinity when the C^2 function v is Lipschitzian.*

Proof. Each $\phi_H^{t_i, t_{i+1}}$ has compact support and therefore bounded derivatives. Hence we can apply Lemma 2.15 with $\psi_1 = 0$, $\ell(x; \eta) = v(x_0) + xy_{N-1} + \sum_{0 \leq i < N} \phi_H^{t_i, t_{i+1}}(x_{i+1}, y_i)$ and $Q(\eta) := -x_{N-1}y_{N-1} + \sum_{0 \leq i < N-1} (x_{i+1} - x_i)y_i$. \square

There does not always exist a GFQI for $L = \varphi_H^{s,t}(dv)$ if H is not compactly supported, even when it satisfies (2.1) and v has as little growth at infinity as possible:

Example 2.17. If the Hamiltonian $H \in C^2([0, T] \times T^*\mathbb{R})$ is given by $H(t, x, y) = x^2 + y^2$, then $\varphi_H^{0,t}(x, y) = (x \cos 2t - y \sin 2t, y \cos 2t + x \sin 2t)$; if $v = 0$, it follows that

$$L := \varphi_H^{0,\pi/4}(dv) = \{0\} \times \mathbb{R}$$

has empty intersection with $\pi^{-1}(x) = \{x\} \times \mathbb{R}$ for $x \neq 0$ and noncompact intersection with $\pi^{-1}(0)$, which prevents L from admitting a GFQI.

When (2.1) is satisfied, however, the Lagrangian $L = \varphi_H^{s,t}(dv)$ does admit a GFQI for small $|s - t|$. Indeed, as $\varphi_H^{s,t}$ is close to the identity, the generating function $\phi_H^{s,t}$ is “small” compared to the quadratic form, hence Lemma 2.15 applies:

Corollary 2.18. *If (2.1) holds then, for each Lipschitzian C^2 function $v : \mathbb{R}^d \rightarrow \mathbb{R}$, there exists a constant α such that for $|t - s| < \alpha$,*

$$S(x; x_0, y_0) = v(x_0) + \phi_H^{s,t}(x, y_0) + xy_0 - x_0y_0$$

is a GFQI for $L = \varphi_H^{s,t}(dv)$.

Remark 2.19. It is essential that v be Lipschitzian: indeed, if $d = 1$, $H(t, x, y) = \frac{1}{2}y^2$ and $v(x) = \frac{1}{3}x^3$, then $\varphi_H^{0,t}(x, y) = (x + ty, y)$ and therefore $\varphi_H^{0,t}(dv) = \{(x + tx^2, x^2)\}$, whose image under the projection π is a half-line for $t \neq 0$.

As the main ingredient in the construction of generating families is the Hamiltonian flow, what matters essentially over a given compact subset of \mathbb{R}^d is the region swept by the Hamiltonian flow; this is the idea of what is called the *property of finite propagation speed* in [CV08], Appendix A:

Proposition 2.20. *Let $[s, t] \subset [0, T]$ and $L = \varphi_H^{s,t}(dv)$. If for any compact subset $K \subset \mathbb{R}^d$, the set*

$$\mathcal{U}_K := \bigcup_{\tau \in [s, t]} \{\tau\} \times \{\varphi_H^{s,\tau}(\varphi_H^{t,s}(\pi^{-1}(K)) \cap dv)\},$$

is compact, then L admits GFQI's in the sense that each $L|_K := L \cap \pi^{-1}(K)$ has a GFQI.

Proof. For any K , let $\tilde{H} = \chi H$, where χ is a compactly supported smooth function on $[0, T] \times T^*\mathbb{R}^d$ equal to 1 in a neighbourhood of \mathcal{U}_K . Then formula (2.3) with $H := \tilde{H}$ gives a GFQI $S_{\tilde{H}}$ for $L|_K = \varphi_H^{s,t}(\pi^{-1}(K) \cap dv)$. \square

Remark 2.21. One can also truncate v , as the effective region for v is $\pi(\varphi_H^{t,s}(\pi^{-1}(K)))$. This may help to localize the minmax.

Condition (2.1) is not required here, provided H is C^2 and such that $\varphi_H^{s,t}$ is defined for all $s, t \in [0, T]$.

Lemma 2.22. *If two families S and S' are quadratic at infinity with $|S - S'|_{C^0} < \infty$, then the associated minmax functions satisfy*

$$|R_S(x) - R_{S'}(x)| \leq |S - S'|_{C^0}.$$

Proof. If $S \leq S'$, then by definition $R_S(x) \leq R_{S'}(x)$. Hence, in general, the inequality $S \leq S' + |S - S'|_{C^0}$ yields $R_S(x) \leq R_{S'}(x) + |S - S'|_{C^0}$. We conclude by exchanging S and S' . \square

Proposition 2.23. *Under the hypotheses of Proposition 2.20 and with the notation of its proof, the Lagrangian submanifold L admits a minmax selector, given by*

$$R(x) = \inf \max S_{\tilde{H}}(x, \eta), \quad \text{if } x \in K \subset \mathbb{R}^d$$

and independent of the truncation \tilde{H} and the subdivision of $[s, t]$ used to define $S_{\tilde{H}}$.

Proof. Let \tilde{H} and \tilde{H}' be two truncations of H on \mathcal{U}_K as in the proof of Proposition 2.20. Let $H^\mu = \mu\tilde{H} + (1-\mu)\tilde{H}'$, $\mu \in [0, 1]$; as the constant c_{H^μ} of (2.1) is uniformly bounded, one can find a subdivision $s = t_0 < t_1 < \dots < t_N = t$ satisfying $|t_i - t_{i+1}| < \delta_{H^\mu}$ for all μ (see Lemma 2.11); if S_μ denotes the corresponding GFQI of $L|_K = L \cap \pi^{-1}(K)$ for $0 \leq \mu \leq 1$ then, by Lemma 2.22, as S_μ depends continuously on μ , so does the minmax $R_{S_\mu}(x)$ for $x \in \pi(L)$.

On the other hand, $R_{S_\mu}(x)$ is a critical value of the map $\eta \mapsto S_\mu(x, \eta)$, and, by (2.5), the set of all such critical values is independent of μ and the subdivision, and depends only on \mathcal{U}_K ; as it has measure zero by Sard's Theorem, $R_{S_\mu}(x)$ is constant for $\mu \in [0, 1]$.

The fact that the critical value $R_S(x)$ itself does not depend on the subdivision is established in Lemma 3.4. \square

Example 2.24. If the base manifold $M = \mathbb{T}^d$, taking its universal covering \mathbb{R}^d , we can consider $v : \mathbb{R}^d \rightarrow \mathbb{R}$ a periodic function and $H : \mathbb{R} \times T^*\mathbb{R}^d \rightarrow \mathbb{R}$ periodic in x . Then in order that $L = \varphi_H^{s,t}(dv)$ admits a GFQI, it is enough to require that the flow $\varphi_H^{s,\tau}$ is well-defined for $\tau \in [s, t]$. Indeed, since dv is compact, $\bigcup_{\tau \in [s, t]} \{\tau\} \times \varphi_H^{s,\tau}(dv)$ is compact, hence the condition of finite propagation speed is satisfied automatically.

Example 2.25. The following hypotheses satisfy the finite propagation speed property:

$$|\partial_y H| \leq C'_H(1 + |x|), \quad |\partial_x H| \leq C_H(1 + |y|),$$

It is a classical condition for the existence and uniqueness of viscosity solutions in \mathbb{R}^d , see [CL87].

2.3. Generalized generating families and minmax in the Lipschitz cases. Already if $d = 1$, $H(t, x, y) = \frac{1}{2}y^2$ and $v(x) = \arctan x$, the Lagrangian submanifold $\varphi_H^{0,t}(dv) = \{(x + \frac{t}{1+x^2}, \frac{1}{1+x^2}) : x \in \mathbb{R}\}$ is not the graph of a function for $t > 0$ large enough, and the minimax of its generating phase $S_t(x; x_0, y_0) = \arctan x_0 + \frac{t}{2}y_0^2 + (x - x_0)y_0$ is not a C^1 function, though it is locally Lipschitzian (see Proposition 2.31 hereafter).

Hence, in order to iterate the minmax procedure, one is led to defining the minmax when the Cauchy datum is a Lipschitzian function. We will use Clarke's generalization of the derivatives of C^1 functions in the Lipschitz setting [Cla83], see the Appendix.

Proposition 2.26. *Under the hypothesis (2.1) and with the notation of Corollary 2.13, if v is only locally Lipschitzian, the family S given by (2.3) generates $L = \varphi_H^{s,t}(\partial v)$ in*

the sense that

$$L = \{(x, \partial_x S(x; \eta)) \mid 0 \in \partial_\eta S(x; \eta)\}.$$

where ∂ denote Clarke's generalized derivative and $\partial v := \{(x, p), p \in \partial v(x)\}$.

Proof. The equation $0 \in \partial_\eta S(x; \eta)$ means that $y_0 \in \partial v(x_0)$ and $y_{i+1} = \partial_{x_{i+1}} \phi_H^{t_i, t_{i+1}}(x_{i+1}, y_i)$, $x_i = \partial_{y_i} \phi_H^{t_i, t_{i+1}}(x_{i+1}, y_i)$ for $0 \leq i < N$, where $x := x_N$ et $\eta = (x_i, y_i)_{0 \leq i < N}$. \square

However, this definition of a generating family is not invariant by fiberwise diffeomorphism, even by the following very simple (and useful) one:

$$(x; (x_i)_{0 \leq i < N}, (y_i)_{0 \leq i < N}) \mapsto (x, (x_{i+1} - x_i, y_i)_{0 \leq i < N}) =: (x, (\xi_i, y_i)_{0 \leq i < N});$$

indeed, it transforms the family S given by (2.3) into

$$S'(x; (\xi_i, y_i)_{0 \leq i < N}) := v\left(x - \sum \xi_i\right) + \sum_{0 \leq i < N} \phi_H^{t_i, t_{i+1}}\left(x - \sum_{i < j < N} \xi_j, y_i\right) + \sum_{0 \leq i < N} \xi_i y_i,$$

for which $\partial_x S'(x; (\xi_i, y_i)_{0 \leq i < N})$ is not a point, but the subset

$$\partial v\left(x - \sum \xi_i\right) + \sum_{0 \leq i < N} \partial_1 \phi_H^{t_i, t_{i+1}}\left(x - \sum_{i < j < N} \xi_j, y_i\right).$$

As often, this difficulty is overcome by finding the right definition*:

Definition 2.27. A Lipschitz family $S : \mathbb{R}^d \times \mathbb{R}^k \rightarrow \mathbb{R}$ is called a *generating family* for $L \subset T^*\mathbb{R}^d$ when

$$L = \{(x, y) \in T^*\mathbb{R}^d \mid \exists \eta \in \mathbb{R}^k : (y, 0) \in \partial S(x, \eta)\}.$$

Lemma 2.28. This definition of a generating family is invariant by fiberwise C^1 diffeomorphisms.

Proof. If $\Phi(x, \eta') = (x, \phi(x, \eta'))$ is a fiberwise diffeomorphism of $\mathbb{R}^d \times \mathbb{R}^k$, and $S' := S \circ \Phi$, then the chain rule (see the Appendix, Lemma A.31) yields

$$\partial S'(x, \eta') = \left\{ \left(y + \zeta \frac{\partial}{\partial x} \phi(x, \eta'), \zeta \frac{\partial}{\partial \eta'} \phi(x, \eta') \right) \mid (y, \zeta) \in \partial S(x, \phi(x, \eta')) \right\};$$

as $\eta' \mapsto \phi(x, \eta')$ is a diffeomorphism, it does follow that the two conditions

$$\exists \eta \in \mathbb{R}^k : (y, 0) \in \partial S(x, \eta) \quad \text{and} \quad \exists \eta' \in \mathbb{R}^k : (y, 0) \in \partial S'(x, \eta')$$

are equivalent. \square

We are now ready to consider GFQI's for the elements of

$$\tilde{\mathcal{L}} := \{L = \varphi(\partial v), \quad v \in C^{\text{Lip}}(\mathbb{R}^d), \varphi \in \text{Ham}_c(T^*\mathbb{R}^d)\} :$$

*But this example exhibits one of the features of the Clarke derivative: the relation $(y, 0) \in \partial S'(x, \eta)$ is definitely not equivalent to $y \in \partial_x S'(x, \eta)$, $0 \in \partial_{\xi_i} S'(x, \eta)$ and $0 \in \partial_{y_i} S'(x, \eta)$.

Proposition 2.29. *If $H : [0, T] \times T^*\mathbb{R}^d \rightarrow \mathbb{R}$ is C^2 and has compact support then, for each $v \in C^{\text{Lip}}(\mathbb{R}^d)$, the generating family of $L = \varphi_H^{s,t}(\partial v) \in \tilde{\mathcal{L}}$ given by (2.3), namely*

$$S(x; \eta) = v(x_0) + \sum_{0 \leq i < N} \phi_H^{t_i, t_{i+1}}(x_{i+1}, y_i) + \sum_{0 \leq i < N} (x_{i+1} - x_i) y_i,$$

where $x_N := x$, $\eta := ((x_i, y_i))_{0 \leq i < N}$, is “quadratic at infinity” in the following sense: let

$$Q(\eta) := -x_{N-1}y_{N-1} + \sum_{0 \leq i < N-1} (x_{i+1} - x_i)y_i,$$

the Lipschitz constant of each $S(x, \cdot) - Q : (T^*\mathbb{R}^d)^N \rightarrow \mathbb{R}$ is bounded, uniformly with respect to x on each compact subset of \mathbb{R}^d .

Hence, for each compact $K \subset \mathbb{R}^k$, if $\theta \in C_c^\infty(\mathbb{R}^d, [0, 1])$ equals 1 in a neighbourhood of 0, there exists a positive constant a_K such that the function

$$(2.7) \quad S_K(x; \eta) = \psi_K(x; \eta) + Q(\eta), \text{ where } \psi_K(x; \eta) := \theta\left(\frac{\eta}{a_K}\right) (S(x, \eta) - Q(\eta)), \quad x \in K$$

is a GFQI of $L_K := L \cap \pi^{-1}(K)$.

In the sequel, unless otherwise specified, we consider families S of the form (2.3) and families S_K of the form (2.7). The advantage is that they are C^1 in x , in which case $\partial S(x, \eta) = \partial_x S \times \partial_\eta S(x, \eta)$. This is not essential, but simplifies our formalism and arguments.

To study the minmax function R_S for such S , we use the extension of classical results in critical point theory to locally Lipschitz functions described in the Appendix.

Proposition 2.30. *The minmax $R_S(x)$ is well-defined and it is a critical value[†] of the map $\eta \mapsto S(x, \eta)$. For each compact subset K of \mathbb{R}^d and each truncation S_K of S of the form (2.7) generating L_K , we have that $R_S(x) = R_{S_K}(x)$ for $x \in K$.*

Proof. By Proposition 2.29, $f(\eta) := S(x, \eta) = \psi(\eta) + Q(\eta)$ with ψ Lipschitzian and Q a nondegenerate quadratic form. Hence, f satisfies the P.S. condition (Appendix, Example A.28). If $c = R_S(x)$ were not a critical value, the flow φ_V^t of Theorem A.29 in the Appendix would deform the descending cycles in $f^{c+\epsilon}$ into descending cycles in $f^{c-\epsilon}$, hence the contradiction $c = \inf \max_\sigma f \leq c - \epsilon$.

To see that $R_S|_K = R_{S_K}$, just notice that every descending cycle σ of $S(x, \cdot)$ or $S_K(x, \cdot)$, $x \in K$, can be deformed into a common descending cycle σ' with $\max S(x, \sigma'(\cdot)) = \max S_K(x, \sigma'(\cdot))$ by using the gradient flow of Q , suitably truncated. \square

Proposition 2.31. *The minmax $R_S(x)$ is a locally Lipschitz function.*

Proof. Let $K \subset \mathbb{R}^d$ be compact. By Proposition 2.30, we have that $R_S|_K = R_{S_K}$, where $S_K : K \times \mathbb{R}^k \rightarrow \mathbb{R}$ writes $S(x, \eta) = \psi_K(x, \eta) + Q(\eta)$ with Q a nondegenerate quadratic form and ψ_K a compactly supported Lipschitz function. Given $x, x' \in K$,

[†]Appendix, Definition A.27.

for all $\epsilon > 0$, there exists a descending cycle $\bar{\sigma}$ such that $\max_{\eta \in \bar{\sigma}} S_K(x, \eta) \leq R_S(x) + \epsilon$; if $\max_{\eta \in \bar{\sigma}} S_K(x', \eta)$ is reached at $\bar{\eta}$, then

$$\begin{aligned} R_S(x') - R_S(x) &\leq S_K(x', \bar{\eta}) - S_K(x, \bar{\eta}) + \epsilon = \psi_K(x', \bar{\eta}) - \psi_K(x, \bar{\eta}) + \epsilon \\ &\leq \text{Lip}(\psi_K)|x - x'| + \epsilon. \end{aligned}$$

If we let $\epsilon \rightarrow 0$ and exchange x and x' , we obtain

$$|R_S(x) - R_S(x')| \leq \text{Lip}(\psi_K)|x - x'|,$$

which proves our result. \square

Proposition 2.32. *The sets $C(x) = \{\eta \mid 0 \in \partial_\eta S(x, \eta), S(x, \eta) = R_S(x)\}$ are compact[‡] and the set-valued map (“correspondence”) $x \mapsto C(x)$ is upper semi-continuous: for every convergent sequence $(x_k, \eta_k) \rightarrow (x, \eta)$ with $\eta_k \in C(x_k)$, one has $\eta \in C(x)$. In other words, the graph $C = \{(x, \eta) \mid \eta \in C(x)\}$ of the correspondence is closed.*

Proof. Let $(x_k, \eta_k) \rightarrow (x, \eta)$ with $\eta_k \in C(x_k)$; then, as S is C^1 in x , $\partial S = \partial_x S \times \partial_\eta S$. Now $\partial S : (x, \eta) \mapsto \partial_x S \times \partial_\eta S$ is upper semi-continuous (Appendix, Proposition A.26), the limit $(\partial_x S(x, \eta), 0)$ of the sequence $(\partial_x S(x_k, \eta_k), 0) \in \partial S(x_k, \eta_k)$ belongs to $\partial S(x, \eta)$, hence $0 \in \partial_\eta S(x, \eta)$; as the continuity of S and R_S implies that $S(x_k, \eta_k) \rightarrow S(x, \eta)$ and $R_S(x_k) \rightarrow R_S(x)$, this proves $\eta \in C(x)$. \square

Lemma 2.33. *Given any $\delta > 0$, there exists an $\epsilon > 0$ such that*

$$R_S(x) = \inf_{\sigma \in \Sigma_\epsilon} \max_{\sigma \cap C_\delta(x)} S(x, \eta)$$

where $\Sigma_\epsilon = \{\sigma \mid \max_\sigma S(x, \eta) \leq R_S(x) + \epsilon\}$ and $C_\delta(x) = B_\delta(C(x))$ denotes the δ -neighborhood of the critical set $C(x)$.

Proof. This is a direct consequence of the deformation lemma (Appendix, Theorem A.30) for $S_x := S(x, \cdot)$: for $\delta > 0$, and $c = R_S(x)$, there exist $\epsilon > 0$ and V such that $\varphi_V^1(S_x^{c+\epsilon} \setminus C_\delta(x)) \subset S_x^{c-\epsilon}$. In particular, we remark that for $\sigma \in \Sigma_\epsilon$, the intersection $\sigma \cap C_\delta(x)$ is non vide, otherwise, the flow φ_V^1 may take σ to a descending cycle $\sigma' = \varphi_V^1(\sigma)$ such that $\max_{\eta \in \sigma'} S_x(\eta) \leq R_S(x) - \epsilon$, contradiction with the definition of minmax. \square

Remark 2.34. When S is C^2 , the S_x ’s are generically Morse functions: indeed, S_x is Morse if and only if x is a regular value of the projection $\pi : L \rightarrow M$, $(x, p) \mapsto x$, whose regular values, by Sard’s theorem and the compactness of $\text{Crit}(S_x)$, form an open set of full measure. In this case, S_x^c is indeed a deformation retract of $S_x^{c+\epsilon}$ for $\epsilon > 0$ small enough, hence $\inf \max$ deserves its name “minmax”, that is, there exists a descending cycle σ such that, $R_S(x) = \max_\sigma S(x, \eta) = \max_{\sigma \cap C(x)} S(x, \eta)$.

Proposition 2.35. *The generalized derivative of R_S satisfies*

$$(2.8) \quad \partial R_S(x) \subset \text{co}\{\partial_x S(x, \eta) \mid \eta \in C(x)\}$$

[‡]Appendix, Example A.28.

Proof. First, we claim that, if R_S is differentiable at \bar{x} , then

$$(2.9) \quad dR_S(\bar{x}) \subset \text{co}\{\partial_x S(\bar{x}, \eta) \mid \eta \in C(\bar{x})\}$$

Take δ and ϵ for \bar{x} as in Lemma 2.33. Consider $K = \overline{B_1(\bar{x})}$, and S_K obtained in Lemma 2.15, one can choose a $\varrho \in (0, 1)$ such that for $x \in B_\varrho(\bar{x})$,

$$|S_K(x, \cdot) - S_K(\bar{x}, \cdot)|_{C^0} \leq \epsilon/4.$$

Now let $y \in \mathbb{R}^d$ and $\lambda < 0$ be small such that $x_\lambda := \bar{x} + \lambda y \in B_\varrho(\bar{x})$ and $\lambda^2 < \epsilon/4$. Then by Lemma 2.33, for each x_λ , there is a descending cycle σ_λ such that

$$\max_{\sigma_\lambda} S(x_\lambda, \eta) \leq R_S(x_\lambda) + \lambda^2$$

then

$$\max_{\sigma_\lambda} S(\bar{x}, \eta) \leq \max_{\sigma_\lambda} S(x_\lambda, \eta) + \frac{\epsilon}{4} \leq R_S(x_\lambda) + \frac{\epsilon}{2} \leq R_S(\bar{x}) + \frac{3\epsilon}{4}$$

and

$$R_S(\bar{x}) \leq \max_{\sigma_\lambda \cap C_\delta(\bar{x})} S(\bar{x}, \eta) = S(\bar{x}, \eta_\lambda), \quad \text{for some } \eta_\lambda \in \sigma_\lambda \cap C_\delta(\bar{x}).$$

Hence we have

$$(2.10) \quad \lambda^{-1}[R_S(x_\lambda) - R_S(\bar{x})] \leq \lambda^{-1}[S(x_\lambda, \eta_\lambda) - S(\bar{x}, \eta_\lambda)] - \lambda$$

$$(2.11) \quad = \langle \partial_x S(x'_\lambda, \eta_\lambda), y \rangle - \lambda,$$

where the last equality is given by the mean value theorem for some x'_λ in the line segment between \bar{x} and x_λ .

Take the lim sup of both sides in the above inequality and let $\delta \rightarrow 0$, we get

$$\langle dR_S(\bar{x}), y \rangle \leq \max_{\eta \in C(\bar{x})} \langle \partial_x S(\bar{x}, \eta), y \rangle, \quad \forall y \in \mathbb{R}^d$$

Note that this implies that $dR_S(\bar{x})$ belongs to the sub-derivative of the convex function $f(y) := \max_{\eta \in C(\bar{x})} \langle \partial_x S(\bar{x}, \eta), y \rangle$ at $v = 0$,[§] for which one can easily calculate

$$\partial f(0) = \text{co}\{\partial_x S(\bar{x}, \eta) : \eta \in C(\bar{x})\}.$$

Thus we get (2.9). In general,

$$\begin{aligned} \partial R_S(x) &= \text{co}\{\lim_{x' \rightarrow x} dR_S(x')\} \subset \text{co}\{\text{co} \lim_{x' \rightarrow x} \{\partial_x S(x', \eta'), \eta' \in C(x')\}\} \\ &\subset \text{co}\{\partial_x S(x, \eta), \eta \in C(x)\} \end{aligned}$$

by the upper-semi continuity of $x \mapsto C(x)$ and the continuity of $\partial_x S$. \square

The formula (2.8) gives us somehow a *generalized graph selector*, while for a classical graph selector, we require that for almost every x ,

$$dR_S(x) = \partial_x S(x, \eta), \quad \text{for some } \eta \in C(x)$$

from which $(x, dR_S(x)) \in L$.

If S is a GFQI of $L = \varphi(dv) \in \mathcal{L}$ for $v \in C^2$, then $S_x := S(x, \cdot)$ is an excellent Morse function for almost every x , in which cases $C(x)$ consists of a single point, hence $\partial R_S(x) = \partial_x S(x, \eta)$ for a unique η , proving that R_S is a true graph selector for L .

[§]Recall that, for a convex function f , the sub-derivative at a point x is the set of ξ such that $f(y) - f(x) \geq \langle \xi, y - x \rangle, \forall y$

Question 2.36. Is the minmax R_S also a true graph selector for $L \in \tilde{\mathcal{L}}$.

Question 2.37. Is it true that, when R_S is differentiable at x , one has $(x, dR_S(x)) \in L$? Here $L \in \mathcal{L}$ or even $\tilde{\mathcal{L}}$.

3. VISCOSITY SOLUTIONS AND MINMAX SOLUTIONS

We look at the solutions of the (H-J) equation, assuming that $H \in C^2([0, T] \times T^*\mathbb{R}^d)$ and $v \in C^{\text{Lip}}(\mathbb{R}^d)$ satisfy the condition of finite propagation speed. Unless otherwise specified, we assume that H has compact support (as a function on $[0, T] \times T^*\mathbb{T}^d$ when H and v are periodic).

3.1. Geometric solution and its minmax selector. Following the classical geometric method for first order partial differential equations, the Hamilton-Jacobi equation is considered to be a hypersurface in the cotangent bundle $T^*(\mathbb{R} \times \mathbb{R}^d)$.

More precisely, let

$$\mathcal{H}(t, x, e, p) =: e + H(t, x, p), \quad (t, x, e, p) \in T^*(\mathbb{R} \times \mathbb{R}^d)$$

and at the moment suppose that the initial function v is C^2 .

Definition 3.1. Let $\varphi_{\mathcal{H}}^s$ denote the Hamiltonian flow of \mathcal{H} , which preserves the levels of \mathcal{H} , and let

$$\Gamma_v = \{ (0, x, -H(0, x, dv(x)), dv(x)) \};$$

then, the *geometric solution* of the Cauchy problem (H-J) is

$$L_{\mathcal{H},v} := \bigcup_{s \in [0, T]} \varphi_{\mathcal{H}}^s(\Gamma_v).$$

It is a Lagrangian submanifold containing the initial isotropic submanifold Γ_v and contained in the hypersurface

$$\mathcal{H}^{-1}(0) = \{ (t, x, e, p) | e + H(t, x, p) = 0 \} \subset T^*(\mathbb{R} \times \mathbb{R}^d).$$

As every Lagrangian submanifold L of $T^*(\mathbb{R} \times \mathbb{R}^d)$ contained in $\mathcal{H}^{-1}(0)$ is locally invariant by $\varphi_{\mathcal{H}}^s$, this geometric solution is in some sense maximal.

Writing $T^*(\mathbb{R} \times \mathbb{R}^d)$ as $T^*\mathbb{R} \times T^*\mathbb{R}^d$, we have $X_{\mathcal{H}} = (1, -\partial_t H, X_H)$, and

$$L_{\mathcal{H},v} = \{ (t, -H(t, \varphi_H^t(dv)), \varphi_H^t(dv)), t \in [0, T] \}$$

where $\varphi_H^t := \varphi_H^{0,t}$ is the Hamiltonian isotopy generated by H .

Lemma 3.2. Formula (2.4) defines a GFQI

$$S : [0, T] \times \mathbb{R}^d \times \mathbb{R}^{2l} \rightarrow \mathbb{R}$$

of $L_{\mathcal{H},v}$.

Proof. For simplicity, we may assume that $T \in (0, \delta_H)$, hence that

$$S : [0, T] \times \mathbb{R}^d \times \mathbb{R}^{2d} \rightarrow \mathbb{R}, \quad S(t, x, x_0, y_0) = v(x_0) + xy_0 + \phi_H^t(x, y_0) - x_0y_0.$$

Let $(x_0, y_0) \in \Sigma_S$, then

$$(\partial_t S(t, x, x_0, y_0), \partial_x S(t, x, x_0, y_0)) = (\partial_t \phi_H^t(x, y_0), \partial_x \phi_H^t(x, y_0)) = (-H(t, x, y(t)), y(t)),$$

where $(x, y(t)) = \varphi_H^t(x_0, y_0)$ with $y_0 = dv(x_0)$.

Hence

$$\{(t, x, \partial_t S(t, x, x_0, y_0), \partial_x S(t, x, x_0, y_0)) | (x_0, y_0) \in \Sigma_S\} = L_{\mathcal{H}, v}$$

□

If there exists a C^1 function $u : [0, T] \times \mathbb{R}^d \rightarrow \mathbb{R}$ such that

$$L = L_{\mathcal{H}, v} = \{(t, x, \partial_x u(t, x), \partial_x u(t, x)) \subset T^*([0, T] \times \mathbb{R}^d)\}$$

we say that L is a 1-graph in $T^*([0, T] \times \mathbb{R}^d)$. In this case, u is a global solution of the Cauchy problem of (H-J) equation. In general, L may be the graph of the derivatives of a multi-valued function.

An equivalent but more economic way to describe the geometric solution is to identify each $\varphi_{\mathcal{H}}^s(\Gamma_v)$ with $\{s\} \times \varphi_H^s(dv)$ by the inverse of the map $(t, x, p) \mapsto (t, x, -H(t, x, p), p)$. In this way, we also call the union

$$L_{H, v} := \bigcup_{t \in [0, T]} \{t\} \times \varphi_H^t(dv) \subset \mathbb{R} \times T^*\mathbb{R}^d$$

a geometric solution.

If we look at the projection of the characteristics, that is, the image of the graph of the solutions $\{(t, \varphi_H^t(x_0, p_0))\}_{t \in [0, T]}$, $(x_0, p_0) \in T^*\mathbb{R}^d$, of Hamilton's equations under the projection

$$\pi : [0, T] \times T^*\mathbb{R}^d \rightarrow \mathbb{R} \times \mathbb{R}^d, \quad (t, x, p) \mapsto (t, x).$$

then L is not a 1-graph when the corresponding characteristics intersect under the projection. Without ambiguity, we will simply say that the characteristics intersect.

Now, as before, we consider more generally the Lipschitz case. Given $v \in C^{\text{Lip}}(\mathbb{R}^d)$, set

$$\Gamma_v = \{(0, x, -H(0, x, p), p) : p \in \partial v(x)\}$$

and similarly

$$\begin{aligned} L_{\mathcal{H}, v} &= \bigcup_{s \in [0, T]} \varphi_{\mathcal{H}}^s(\Gamma_v) = \left\{ \left(t, -H(t, \varphi_H^t(x, p)), \varphi_H^t(x, p) \right) : p \in \partial v(x), t \in [0, T] \right\} \\ L_{H, v} &= \bigcup_{t \in [0, T]} \{t\} \times \varphi_H^t(\partial v) := \bigcup_{t \in [0, T]} \{t\} \times \{\varphi_H^t(x, p) : p \in \partial v(x)\} \end{aligned}$$

where ∂ is Clarke's generalized derivative. We call $L_{\mathcal{H}, v}$ or $L_{H, v}$ *generalized geometric solutions*. They are also generated by the GFQI given by formula (2.4) on page 5.

Definition 3.3. For any time $0 \leq s < t \leq T$, we define the minimax operator[¶]

$$R_H^{s,\tau} : C^{\text{Lip}}(\mathbb{R}^d) \rightarrow C^{\text{Lip}}(\mathbb{R}^d), \quad \tau \in [s, t]$$

for the (H-J) equation as

$$R_H^{s,\tau} v(x) = \inf_{\eta} \max S(\tau, x, \eta)$$

where $S : [s, t] \times \mathbb{R}^d \times \mathbb{R}^k \rightarrow \mathbb{R}$ is given by (2.4).

For completeness, without referring to the uniqueness theorem for GFQI's, we give a proof that the minmax is well-defined independently of the subdivisions.

Lemma 3.4. *The minmax $R_S(x) = \inf \max S(x, \eta)$ given by (2.3) or (2.4) is independent of the subdivision of time in the construction of S .*

Proof. First assume $t - s < \delta_H$; given $\tau \in (s, t)$, consider the family of subdivisions $\zeta_\mu := \{s \leq s + \mu(\tau - s) < t\}$; then,

$$S_\mu(x; x_0, y_0, x_1, y_1) = v(x_0) + \phi_H^{s, s+\mu(\tau-s)}(x_1, y_0) + (x_1 - x_0)y_0 + \phi_H^{s+\mu(\tau-s), t}(x, y_1) + (x_2 - x_1)y_1,$$

where $x_2 := x$, is the generating family defined by (2.3) and associated to ζ_μ , $\mu \in (0, 1]$. The function S_μ is continuous in μ and the minmax $R_{S_\mu}(x)$ is the critical value of the map $\eta \mapsto S_\mu(x; \eta)$ with $\eta := (x_0, y_0, x_1, y_1)$. By 2.5, the set of all such critical values is independent of μ ; as it has measure zero by Sard's Theorem, R_{S_μ} is constant for $\mu \in [0, 1]$. In particular, letting $x'_1 := x_1 - x_0$ and $y'_0 = y_0 - y_1$, we get

$$S_0(x; x_0, y_0, x_1, y_1) = S_0(x; (x_0, y_1, x'_1, y'_0)) = v(x_0) + \phi_H^{s,t}(x_2, y_1) + (x_2 - x_0)y_1 + x'_1 y'_0.$$

It is obtained by adding the quadratic form $x'_1 y'_0$ to

$$S(x; x_0, y_1) = v(x_0) + \phi_H^{s,t}(x_2, y_1) + (x_2 - x_0)y_1,$$

which is the generating family related to ζ_0 . We conclude that $R_S(x) = R_{S_0}(x) = R_{S_1}(x)$.

In general, given any two subdivisions ζ', ζ'' of $[s, t]$ with^{||} $|\zeta'|, |\zeta''| < \delta_H$, denote by $\zeta = \zeta' \cup \zeta'' = \{s = t_0 < \dots < t_n = t\}$ the subdivision obtained by collecting the points in ζ' and ζ'' . If t_j is not contained in ζ' , we consider the family of subdivisions

$$\zeta_\mu(j) = \{t_0 < t_{j-1} \leq t_{j-1} + \mu(t_j - t_{j-1}) < t_{j+1} < \dots < t_n\}, \quad \mu \in [0, 1]$$

The same argument as before shows that the minmax relative to $\zeta_0(j)$ and $\zeta_1(j)$ are the same. Continuing this procedure, we get that the minmax relative to ζ' and ζ are the same, and the same holds for ζ'' and ζ . Therefore the minmax with respect to ζ' and ζ'' are the same. \square

Lemma 3.5. *If $v \in C^2 \cap C^{\text{Lip}}(\mathbb{R}^d)$, then $R_H^{0,t} v(x)$ verifies the (H-J) equation almost everywhere.*

Proof. This is a direct consequence of the fact that S is a GFQI of $L_{\mathcal{H},v}$ and the minmax is a graph selector in this case. \square

[¶]The inclusion $R_H^{s,\tau}(C^{\text{Lip}}(\mathbb{R}^d)) \subset C^{\text{Lip}}(\mathbb{R}^d)$ is proven in Proposition 3.17 p. 20.

^{||}For a subdivision $\zeta = \{t_0 < \dots < t_n\}$, we let $|\zeta| := \max_i |t_{i+1} - t_i|$.

In general, for a Lipschitzian initial function, we do not know whether the minmax verifies the equation almost everywhere or not. But in view of the estimation of generalized derivatives in Proposition 2.35, we still call $R_H^{s,t}v(x)$ the (generalized) *minmax solution* of the Cauchy problem of (H-J) equation.

Lemma 3.6. *If v is C^2 with bounded second derivative, then there exists an $\epsilon > 0$ such that for $t \in [0, \epsilon)$, the minmax $R_H^{0,t}v(x)$ is C^2 .*

Proof. We will show that, there exists an $\epsilon > 0$, such that for $t \in (0, \epsilon)$, the characteristics beginning from the graph dv do not intersect. More precisely, the map $f_t : x_0 \mapsto X_0^t(x_0, dv(x_0))$ is a diffeomorphism. Indeed, for t small enough,

$$\text{Lip}(f_t - Id) \leq \text{Lip}(\alpha_0^t - Id)(1 + \text{Lip}(dv)) < 1$$

where α_0^t and c_H are defined in Lemma 2.11. This in turn means that the projection map $L = \varphi_H^t(dv) \rightarrow \mathbb{R}^d$, $(x, p) \mapsto x$ is a diffeomorphism, hence $L = \{x, dR_H^{0,t}v(x)\}$, from which we obtain that $R_H^{0,t}v(x)$ is C^2 . \square

3.2. Viscosity solutions.

Definition 3.7. A function $u \in C^0((0, T) \times \mathbb{R}^d)$ is called a *viscosity subsolution* (resp. *supersolution*) of

$$\partial_t u + H(t, x, \partial_x u) = 0$$

when it has the following property: for every $\psi \in C^1((0, T) \times \mathbb{R}^d)$ and every point (t, x) at which $u - \psi$ attains a local maximum (resp. minimum), one has

$$\partial_t \psi + H(t, x, \partial_x \psi) \leq 0, \quad (\text{resp. } \geq 0).$$

The function u is a *viscosity solution* if it is both a viscosity subsolution and supersolution.

We remark that one can replace C^1 test functions ψ by C^∞ test functions in the definition. Obviously, a classical C^1 solution is a viscosity solution. Indeed, we have

Lemma 3.8. *A viscosity solution verifies the equation wherever it is differentiable.*

Proof. Just observe that, if $\psi - u$ reaches a local minimum (resp. maximum) at a point (t, x) where u is differentiable, then $d\psi(t, x) = du(t, x)$ and use the definition. \square

The existence of viscosity solutions is ensured by the so-called “vanishing viscosity method” at the origin of the name “viscosity”. The approach is to consider the approximate problem

$$(HJ_\epsilon) \quad \begin{cases} \partial_t u^\epsilon + H(t, x, \partial_x u^\epsilon) = \epsilon \Delta u^\epsilon \\ u^\epsilon(0, x) = v(x) \end{cases}$$

for $\epsilon > 0$. This quasilinear parabolic Cauchy problem turns out to have a smooth solution u^ϵ , as the viscosity term $\epsilon \Delta$ regularizes the Hamilton-Jacobi equation.

Proposition 3.9 ([CEL84]). *The sequence $\{u^\epsilon\}_\epsilon$ tends to the viscosity solution of the (H-J) problem as $\epsilon \rightarrow 0$.*

Uniqueness follows at once from the following estimate:

Proposition 3.10 ([CL87]). *If u_1 and u_2 are viscosity solutions of the Hamilton-Jacobi equation, then*

$$\sup_{x \in \mathbb{R}^d} (u_1(t, x) - u_2(t, x))^+ \leq \sup_{x \in \mathbb{R}^d} (u_1(0, x) - u_2(0, x))^+.$$

This proposition gives more than uniqueness: it provides a monotonicity property for viscosity solutions with respect to the initial condition: if $u_1(0, \cdot) \leq u_2(0, \cdot)$, then $u_1 \leq u_2$.

Theorem 3.11 ([CL87]). *If $v \in C^{\text{Lip}}(\mathbb{R}^d)$ and $H \in C_c^2([0, T] \times T^*\mathbb{R}^d)$, then there exists a unique viscosity solution of the Cauchy problem of the Hamilton-Jacobi equation. Moreover, this solution is globally Lipschitz.*

A notable feature of the viscosity solution, is that it is *Markovian*, meaning that, if

$$J_s^t : C^{\text{Lip}}(\mathbb{R}^d) \rightarrow C^{\text{Lip}}(\mathbb{R}^d)$$

denotes the viscosity solution operator (for a fixed Hamiltonian) which to v associates the time t of the solution equal to v at time s , then the “two-parameter groupoid” property

$$J_\tau^t = J_s^t \circ J_\tau^s$$

is satisfied. This follows easily from uniqueness.

The following Proposition summarizes a well-studied case when the Hamiltonian is convex in p , one can refer to [Jou93, Cha90, WEI13].

Proposition 3.12. *Assume $H \in C^2([0, T] \times T^*\mathbb{R}^d)$ strictly convex in p , equal to $|p|^2$ off a compact set, and $v \in C^{\text{Lip}}(\mathbb{R}^d)$. Then the minmax solution is reduced to a min, and it possesses the “semi-group” property with respect to time, that is*

$$R_0^t v(x) = R_s^t \circ R_0^s v(x), \quad 0 \leq s \leq t$$

Proof. Our hypotheses imply that there exists a constant $\epsilon_H > 0$ such that, for $0 < t - s < \epsilon_H$, the Hamiltonian diffeomorphism φ_s^t of H has a classical generating function $\psi_s^t(X, x)$ in the sense that $((x, y), (X, Y)) \in \text{Graph}(\varphi)$ if and only if

$$\begin{cases} Y = \partial_X \psi(X, x) \\ y = -\partial_x \psi(X, x). \end{cases}$$

Therefore, for any $0 \leq s < t \leq T$, the subset $L = \varphi_s^t(\partial v)$ has the generating family

$$F_s^t(x; (x_i)_{0 \leq i \leq j}) = v(x_0) + \Psi_s^t(x, (x_i)) := v(x_0) + \sum_{0 \leq i \leq j} \psi_{\tau_i}^{\tau_{i+1}}(x_{i+1}, x_i)$$

where $x_{j+1} := x$ and $\{s = \tau_0 < \tau_1 < \dots < \tau_{j+1} = t\}$ is a subdivision of $[s, t]$ such that $|\tau_i - \tau_{i+1}| < \epsilon_H$, $0 \leq i \leq j$. Up to diffeomorphism, F_s^t is quadratic of index 0 at infinity, since H equals $|p|^2$ off a compact subset. Thus the minmax is reduced to min:

$$R_s^t v(x) = \min_{(x_i)} F_s^t(x, (x_i))$$

Note that by Lemma 3.4, R_s^t is independent of the choice of the subdivision, hence

$$\begin{aligned} R_s^t \circ R_0^s v(x) &= \min_{(x_i)} (R_0^s v(x_0) + \Psi_s^t(x, (x_i))) \\ &= \min_{(x_i)} \left(\min_{(x'_j)} (v(x'_0) + \Psi_0^s(x_0, (x'_j))) + \Psi_s^t(x, (x_i)) \right) \\ &= \min_{(x_i), (x'_j)} (v(x'_0) + \Psi_0^t(x, (x_i), (x'_j))) = R_0^t v(x) \end{aligned}$$

□

Remark 3.13. The hypothesis that $H = |p|^2$ at infinity can be generalized to the case where the condition of finite propagation speed is satisfied. The min solution operator R_0^t is a finite dimensional “discretization” of the Lax-Oleinik semi-group in weak KAM theory, defined by

$$T_s^t v(x) = \inf_{\gamma(t)=x} \left\{ v(\gamma(s)) + \int_s^t L(t, \gamma(t), \dot{\gamma}(t)) dt \right\}$$

where L is the Legendre tranform of H with respect to the p variable, and the inf is taken over all absolutely continuous paths $\gamma : [s, t] \rightarrow \mathbb{R}^d$.

Theorem 3.14 ([Jou93]). *The min solution $R_0^t v(x)$ is the viscosity solution of the Cauchy problem (H-J).*

A generalization of Theorem 3.14 can be made from convex Hamiltonians to the convex-concave type Hamiltonians. Consider the Cauchy problem (H-J) with

$$(SP) \quad H(t, x, p) = H_1(t, x_1, p_1) + H_2(t, x_2, p_2), \quad v(x) = v_1(x_1) + v_2(x_2)$$

where $(x, p) = (x_1, x_2, p_1, p_2) \in T^*\mathbb{R}^d$, and H_1 and H_2 are strictly convex and concave in p respectively. We may assume that each v_i is globally Lipschitz and H_i verifies the condition of finite propagation speed.

Let $S_{(H,v)}$ denote the GFQI related to (H, v) , i.e. the GFQI of the Lagrangian submanifold $\varphi_H^{0,t}(\partial v)$, then we get a GFQI of $\varphi_H^{0,t}(\partial v)$

$$S_{(H,v)}(t, x, \xi) = \sum_{i=1}^2 S_{(H_i, v_i)}(t, x_i, \xi_i)$$

Due to the product formulas for both the minmax and the viscosity solution (see for example Lemma 2.39, 2.40 in [WEI13]), we have

Proposition 3.15. *Suppose that H and v satisfy (SP), then the minmax solution $R_H^t v(x) = \sum_{i=1}^2 R_{H_i}^t v(x_i)$ is the viscosity solution of the (H-J) equation.*

3.3. Iterated minmax and viscosity solution. In contrast to the case of convex Hamiltonians, where the minmax is reduced to a min and provides the viscosity solution, for general non-convex Hamiltonians, the minmax and the viscosity solution may differ: see [Vit96, VO95, BC11] for counterexamples, and also [Che] for a very nice geometric illustration of the fact that the viscosity solution is not necessarily contained in the geometric solution.

Particularly, in [Vit96], the author pointed out without proof that the minmax does not provide a semi-group as a consequence of not being viscosity. We will make this point clear by showing that the semi-group property is a sufficient condition for the minmax to be viscosity.

Proposition 3.16. *Given v , the minmax $R_H^{0,t}v(x)$ is the viscosity solution of the Cauchy problem (H-J) if and only if it has the semi-group property with respect to time, that is,*

$$R_H^{0,t}v(x) = R_H^{s,t} \circ R_H^{0,s}v(x), \quad 0 \leq s < t \leq T.$$

Proof. Suppose $R_0^t v(x) := R_H^{0,t}v(x)$ possesses the semi-group property, we first show that $R_0^t v(x)$ is a viscosity subsolution. For any (t, x) , let ψ be a C^2 function such that $\psi(s, y) =: \psi_s(y) \geq R_0^s v(y)$, with equality at (t, x) . It is enough to consider ψ in a neighborhood of (t, x) , where it has bounded second derivative. Then

$$(3.12) \quad \psi_t(x) = R_\tau^t \circ R_0^\tau v(x) \leq R_\tau^t \psi_\tau(x).$$

By Lemma 3.6, for $t - \tau > 0$ small enough, the characteristics originating from $d\psi_\tau$ do not intersect: let $(x_t, y_t) = \varphi_\tau^t(x_\tau, \partial_x \psi_\tau(x_\tau))$, where φ denotes the Hamiltonian flow of H , then the map $p : (x_\tau, \partial_x \psi_\tau(x_\tau)) \mapsto x_t$ is a diffeomorphism. Therefore $R_\tau^t \psi_\tau(x)$ is a classical C^2 solution of the (H-J) equation. Hence

$$(3.13) \quad R_\tau^t \psi_\tau(x) = \psi_\tau(x) - \int_\tau^t H(s, x, \partial_x R_\tau^s \psi_\tau(x)) ds$$

Moreover, since $(x, \partial_x R_\tau^t \psi_\tau(x)) = \varphi_\tau^t \circ p^{-1}(x)$, we get that $\partial_x R_\tau^t \psi_\tau(x)$ is continuous in τ .

Subtract (3.13) into (3.12), move $\psi_t(x)$ to the RHS, divide both side by $t - \tau$ and let $\tau \rightarrow t$, we get

$$0 \leq -\partial_t \psi_t(x) - H(t, x, \partial_x \psi_t(x))$$

from which we get a subsolution by definition. Similarly, we can prove that $R_0^t v(x)$ is a viscosity supersolution.

Conversely, if $R_0^t v(x)$ is the viscosity solutions, by the uniqueness of viscosity solution, it satisfies the semi-group property. \square

As a direct consequence, we get Theorem 3.14 since the min solutions form a semi-group (Proposition 3.12).

We remark that Proposition 3.16 does not essentially depend on the variational formulation of the minmax. P. Bernard has shown a more general statement, containing the minmax as a special case. It says that if an operator has the semi-group property, monotonicity, and compatibility with the (H-J) equation, then it is the viscosity solution operator, see Proposition 20 in [Ber11].

To compensate the defect of the minmax for not being a semi-group, an idea due to M. Chaperon is to replace the “minmax” by some “iterated minmax”. Roughly speaking, an iterated minmax is obtained by dividing a given time interval into small pieces and take the minmax step by step. This is a priori a discrete semi-group with respect to the points of the subdivision. We are going to show that, as the steps of

the subdivision go to zero, the iterated minmax will converge to a real semi-group, and therefore to the viscosity solution.

In the following, we denote the Lipschitz constant of a global Lipschitz function f by $\|\partial f\|$ and $|\cdot|_K$ denotes the maximum norm on a compact set K .

Lemma 3.17. *Assuming $H \in C_c^2([0, T] \times T^*\mathbb{R}^d)$ and $v \in C^{\text{Lip}}(\mathbb{R}^d)$, we have the following estimations:*

1) $R_H^{s,t}$ defines an operator from $C^{\text{Lip}}(\mathbb{R}^d)$ to $C^{\text{Lip}}(\mathbb{R}^d)$, with Lipschitz constant

$$\|\partial(R_H^{s,t}v)\| \leq \|\partial v\| + \|\partial_x H\| |t - s|$$

2) For any $0 \leq s < t_i \leq T$, $i = 1, 2$,

$$|R_H^{s,t_1}v(x) - R_H^{s,t_2}v(x)| \leq |t_1 - t_2| \max_{t \in [t_1, t_2]} |H(t, x, \cdot)|_Y$$

where $Y = \{y : |y| \leq \|\partial v\| + \|\partial_x H\| \max_i |t_i - s|\}$.

3) Let H^0 and H^1 be two Hamiltonians, then

$$|R_{H^0}^{s,t}v - R_{H^1}^{s,t}v|_{C^0} \leq |t - s| \max_{\tau \in [s, t], y \in Y'} |(H^0 - H^1)(\tau, \cdot, y)|_{C^0}$$

where $Y' = \{y : |y| \leq \|\partial v\| + \max_i \|\partial_x H^i\| |t - s|\}$.

4) If $v^0, v^1 \in C^{\text{Lip}}(\mathbb{R}^d)$ and K is a compact set in \mathbb{R}^d , then there exists a bounded subset $\tilde{K} \subset \mathbb{R}^d$ which depends on $K \times [0, T]$ and the constants $\|\partial v^i\|$, such that

$$(3.14) \quad |R_H^{s,t}v^0 - R_H^{s,t}v^1|_K \leq |v^0 - v^1|_{\tilde{K}}.$$

Proof. The proof is based on Proposition 2.35 with some variation on the original variable x , which can be either $t \in [0, T]$, or $x \in \mathbb{R}^d$ or some parameter λ for the generating family constructed as below.

For simplicity, we may first assume that $|t - s| < \delta_H$ so that

$$S^{s,t}(x, x_0, y_0) = v(x_0) + \phi_H^{s,t}(x, y_0) + xy_0 - x_0y_0$$

Let $(x(\tau), y(\tau))$ denote the Hamiltonian flow, and $C(x)$ be the critical set defined in Proposition 2.35.

1) For $(x_0, y_0) \in C(x)$, we have

$$\partial_x S^{s,t}(x, x_0, y_0) = \partial_x \phi_H^{s,t}(x, y_0) + y_0 = y(t)$$

where

$$y(t) = y_0 - \int_s^t \partial_x H(\tau, x(\tau), y(\tau)) d\tau, \quad y_0 \in \partial v(x_0)$$

Hence by (2.8),

$$\partial R_H^{s,t}v(x) \subset \text{co}\{y(t), y_0 \in \partial v(x_0)\}$$

thus

$$\|\partial(R_H^{s,t}v)\| \leq \|\partial v\| + \|\partial_x H\| |t - s|.$$

2) For $(x_0, y_0) \in C(x)$, we have

$$\partial_t S^{s,t}(x, x_0, y_0) = \partial_t \phi_H^{s,t}(x, y_0) = -H(t, x, y(t))$$

Hence

$$\partial_t R_H^{s,t} v(x) \subset \text{co}\{-H(t, x, y(t)), y_0 \in \partial v(x_0)\}$$

from which

$$|R_H^{s,t_1} v(x) - R_H^{s,t_2} v(x)| \leq |t_1 - t_2| \max_{t \in [t_1, t_2], y \in Y} |H(t, x, y)|$$

where $Y = \{y : |y| \leq \|\partial v\| + \|\partial_x H\| \max_i |t_i - s|\}$.

3) Let $H^\lambda = (1 - \lambda)H^0 + \lambda H^1$, $\lambda \in [0, 1]$, and let $S_\lambda^{s,t}$ be the corresponding generating families. Fix λ , for (x_0, y_0) in the critical set $C^\lambda(x)$ corresponding to H^λ ,

$$\partial_\lambda S_\lambda^{s,t}(x, x_0, y_0) = \partial_\lambda \phi_{H^\lambda}^{s,t}(x, x_0, y_0) = \int_s^t (H^0 - H^1)(\tau, x^\lambda(\tau), y^\lambda(\tau)) d\tau.$$

Hence

$$\partial_\lambda R_{H^\lambda}^{s,t} v(x) \subset \text{co}\left\{\int_s^t (H^0 - H^1)(\tau, x^\lambda(\tau), y^\lambda(\tau)) d\tau, y_0 \in \partial v(x_0)\right\}$$

from which

$$\begin{aligned} |R_{H^0}^{s,t} v(x) - R_{H^1}^{s,t} v(x)| &\leq \int_0^1 \int_s^t |H^0 - H^1|(\tau, x^\lambda(\tau), y^\lambda(\tau)) d\tau d\lambda \\ &\leq |t - s| \max_{\tau \in [s, t], y \in Y'} |(H^0 - H^1)(\tau, \cdot, y)|_{C^0} \end{aligned}$$

where $Y' = \{y : |y| \leq \|\partial v\| + \max_i \|\partial_x H^i\| |t - s|\}$.

4) Let $v^\lambda = (1 - \lambda)v^0 + \lambda v^1$, $\lambda \in [0, 1]$ and $S_\lambda^{s,t}$ denotes the corresponding generating families, then $\partial_\lambda S_\lambda^{s,t}(x, x_0, y_0) = v^1(x_0) - v^0(x_0)$,

$$\partial_\lambda R_H^{s,t} v^\lambda(x) \subset \text{co}\{v^1(x_0) - v^0(x_0) : (x_0, y_0) \in C^\lambda(x)\}$$

with $C^\lambda(x) \subset \{(x_0, y_0) : |x_0| \leq |x| + T \|\partial_y(H|_{\{y \in Y\}})\|\}$, $Y := \{y : |y| \leq \|\partial v\| + T \|\partial_x H\|\}$.

If we take $\tilde{K} = \{x_0 : |x_0| \leq |x|_K + T \|\partial_y(H|_{\{y \in Y\}})\|\}$, we obtain

$$|R_H^{s,t} v^0 - R_H^{s,t} v^1|_K \leq |v^0 - v^1|_{\tilde{K}}$$

In general, the above results follow from the fact that the critical set $C(x)$ defines the Hamiltonian flow $(x(\tau), y(\tau))_{s \leq \tau \leq t}$ for any $0 \leq s < t \leq T$. \square

Remark 3.18. The estimates in the proposition, more subtle than needed, precisely reveal that being with finite propagation speed is enough to define the minmax function.

Now given any compact subset $K \subset \mathbb{R}^d$, we consider $(t, x) \in [0, T] \times K$.

Given a subdivision $\zeta_n = \{0 = t_0 < t_1 < \dots < t_n = T\}$ of $[0, T]$, for each $s \in [0, T]$, we associate to it a number $m(\zeta_n, s)$, depending on ζ_n :

$$m(\zeta_n, s) := i, \quad \text{if } t_i \leq s < t_{i+1}.$$

For simplicity, fixing a subdivision, we may abbreviate $m(\zeta_n, s)$ as $m(n, s)$.

Definition 3.19. The *iterated minmax solution operator* for the (H-J) equation with respect to a subdivision ζ_n is defined as follows: for $0 \leq s' < s \leq T$,

$$R_{H,\zeta_n}^{s',s} := R_H^{t_{m(n,s)},s} \circ \dots \circ R_H^{s',t_{m(n,s')]+1}.$$

When the Hamiltonian H is fixed, we may abbreviate our notation $R_H^{s,t}$ as R_s^t , and the iterated minmax as

$$(3.15) \quad R_{s',n}^s := R_{t_{m(n,s)}}^s \circ \dots R_{s'}^{t_{m(n,s')]+1}.$$

for which we call it a n -step minmax, with a subdivision indicated.

Define the length of ζ_n by $|\zeta_n| := \max_i |t_i - t_{i+1}|$. Suppose that $\{\zeta_n\}_n$ is a sequence of subdivisions of $[0, T]$ such that $|\zeta_n|$ tends to zero as n goes to infinity, and $\{R_{0,n}^s v(x)\}_n$ be the corresponding sequence of iterated minmax solutions for an initial function $v \in C^{\text{Lip}}(\mathbb{R}^d)$.

Lemma 3.20. *The sequence of functions $u_n(s, x) := R_{0,n}^s v(x)$ is equi-Lipschitz and uniformly bounded for $(s, x) \in [0, T] \times K$.*

Proof. By Lemma 3.17, one can verify that

$$\begin{aligned} \|\partial(R_{0,n}^s v)\| &\leq \|\partial v\| + T\|\partial_x H\|, \\ |R_{0,n}^s v - R_{0,n}^t v|_K &\leq |H|_K |s - t|, \quad s, t \in [0, T]. \end{aligned}$$

where $\mathcal{K} := \{(t, x, y) : t \in [0, T], x \in K, |y| \leq \|\partial v\| + T\|\partial_x H\|\}$.

In particular, taking $t = 0$, we get

$$|R_{0,n}^s v|_K \leq |v|_K + T|H|_K, \quad s \in [0, T]$$

□

It follows immediately from the Arzela-Ascoli Theorem that $\{R_{0,n}^s v(x)\}_n$ has uniformly convergent subsequences on $[0, T] \times K$. Fixing a convergent subsequence $\{R_{0,n_k}^s v(x)\}_k$, we write its limit as

$$\bar{R}_0^s v(x) := \lim_{k \rightarrow \infty} R_{0,n_k}^s v(x).$$

Remember that $\bar{R}_0^s v(x)$ depends a priori on the specified subsequence of subdivisions $\{\zeta_{n_k}\}_k$, which itself depends on the given initial function v and the given subdivisions $\{\zeta_n\}_n$. We define the related limit operator for R_{s',n_k}^s with respect to the fixed subsequence of subdivisions $\{\zeta_{n_k}\}_k$: for any time $0 \leq s' < s \leq T$,

$$\bar{R}_{s'}^s := \lim_{k \rightarrow \infty} R_{s',n_k}^s, \quad \text{if the limit exists}$$

Lemma 3.21. *We have*

$$(3.16) \quad \bar{R}_0^s v(x) = \lim_{k \rightarrow \infty} R_{s',n_k}^s \circ \bar{R}_0^{s'} v(x) = \bar{R}_{s'}^s \circ \bar{R}_0^{s'} v(x), \quad \forall 0 \leq s' < s \leq T.$$

Proof. For brevity, we omit the subindex k of n_k .

We first claim that

$$(3.17) \quad \bar{R}_0^s v(x) = \lim_{n \rightarrow \infty} R_{0,n}^{t_{m(n,s)}} v(x)$$

Indeed,

$$\begin{aligned} |R_{0,n}^s v(x) - R_{0,n}^{t_{m(n,s)}} v(x)| &= |R_{t_{m(n,s)}}^s \circ R_{0,n}^{t_{m(n,s)}} v(x) - R_{0,n}^{t_{m(n,s)}} v(x)| \\ &\leq |H|_{\mathcal{K}}(s - t_{m(n,s)}) \leq |H|_{\mathcal{K}}|\zeta_n| \rightarrow 0, \quad n \rightarrow \infty. \end{aligned}$$

Now we suppose the uniform convergence of $\{R_0^{t_{m(n,s)}} v(x)\}_n$ on a bit larger set $\tilde{K} \times [0, T]$ where $\tilde{K} \supset K$ as defined in 4) of Lemma 3.17. Then for any $\epsilon > 0$, there exists N large enough such that for any $i, j > N$,

$$|R_{0,i}^{t_{m(i,s)}} v - R_{0,j}^{t_{m(j,s)}} v|_{\tilde{K}} < \epsilon, \quad \forall s \in [0, T]$$

Hence

$$\begin{aligned} |R_{t_{m(i,s')},i}^{t_{m(i,s)}} \circ R_{0,j}^{t_{m(j,s')}} v - R_{0,i}^{t_{m(i,s)}} v|_K &= |R_{t_{m(i,s')},i}^{t_{m(i,s)}} \circ R_{0,j}^{t_{m(j,s')}} v - R_{t_{m(i,s')},i}^{t_{m(i,s)}} \circ R_{0,i}^{t_{m(i,s')}} v|_K \\ &\leq |R_{0,j}^{t_{m(j,s')}} v - R_{0,i}^{t_{m(i,s')}} v|_{\tilde{K}} < \epsilon. \end{aligned}$$

Let j go to infinity, we get

$$|R_{t_{m(i,s')},i}^{t_{m(i,s)}} \circ \bar{R}_0^{s'} v - R_{0,i}^{t_{m(i,s)}} v|_K < \epsilon, \quad i > N$$

Thus

$$\bar{R}_0^s v(x) = \lim_{i \rightarrow \infty} R_{t_{m(i,s')},i}^{t_{m(i,s)}} \circ \bar{R}_0^{s'} v(x).$$

We conclude by verifying the following, which is similar to (3.17),

$$\lim_{i \rightarrow \infty} R_{s',i}^s \circ \bar{R}_0^{s'} v(x) = \lim_{i \rightarrow \infty} R_{t_{m(i,s')},i}^{t_{m(i,s)}} \circ \bar{R}_0^{s'} v(x).$$

□

Proposition 3.22. $\bar{R}_0^s v(x)$ is the viscosity solution of the $(H-J)$ problem.

Proof. We first show that it is a viscosity subsolution. For any (t, x) , suppose ψ a C^2 function defined in a neighborhood of (t, x) , having bounded second derivative and such that $\psi(s, y) =: \psi_s(y) \geq \bar{R}_0^s v(y)$, with equality at (t, x) ,

$$\psi_t(x) = \bar{R}_0^t v(x) = \lim_{k \rightarrow \infty} R_{\tau, n_k}^t \circ \bar{R}_0^\tau v(x) \leq \lim_{k \rightarrow \infty} R_{\tau, n_k}^t \psi_\tau(x) = R_\tau^t \psi_\tau(x)$$

the last equality holds for $t - \tau$ small enough, where the characteristics originating from $d\psi_\tau$ do not intersect, hence the iterated minmax is nothing but the 1-step minmax which is the classical C^2 solution. We conclude by applying the same argument in Proposition 3.16. The proof that $\bar{R}_0^s v(x)$ is a supersolution is similar. □

For given H and v , we say that *the limit of iterated minmax solutions exists* in $[s, t]$, if for any sequence of subdivision $\{\zeta_n\}_{n \in \mathbb{N}}$ of $[s, t]$ such that $|\zeta_n| \rightarrow 0$ as $n \rightarrow \infty$, the related sequence of iterated minmax solutions $\{R_{H, \zeta_n}^{s, \tau} v(x)\}_{n \in \mathbb{N}}$, $(\tau, x) \in [s, t] \times \mathbb{R}^d$ converges uniformly on compact subsets to a limit which is independent of the choice of subdivisions, then, without ambiguity, we denote this limit also by $\bar{R}_H^{s, \tau} v(x)$. As before, in the case where H is specified once and for all, we may write the iterated minmax solution and its limit by $R_{s, n}^\tau v(x)$ and $\bar{R}_s^\tau v(x)$ respectively.

We can now prove our main Theorem

Theorem 3.23. *Suppose $H \in C_c^2([0, T] \times T^*\mathbb{R}^d)$ and $v \in C^{\text{Lip}}(\mathbb{R}^d)$, then for the Cauchy problem of the Hamilton-Jacobi equation*

$$\begin{cases} \partial_t u + H(t, x, \partial_x u) = 0, & t \in (0, T] \\ u(x, 0) = v(x), & x \in \mathbb{R}^d. \end{cases}$$

the limit of iterated minmax solutions exists and coincides with the viscosity solution.

Proof. For any compact subset K and $(t, x) \in [0, T] \times K$, and given any sequence of subdivisions $\{\zeta_n\}_n$, setting $u_n(t, x) = R_{0, \zeta_n}^t v(x)$, we have proved in Proposition 3.22 that any convergent subsequence of $\{u_n\}_n$ converges uniformly on $[0, T] \times K$ to the viscosity solution. Now, by Lemma 3.20 and the Arzela-Ascoli theorem, the sequence of functions u_n takes its values in a compact subset of $C^0([0, T] \times K)$, hence it converges to the viscosity solution. \square

APPENDIX. LIPSCHITZ CRITICAL POINT THEORY

We will give a brief review on Lipschitz critical point theory, extracted from Appendix B in [WEI13]. Let us consider a real locally Lipschitz function f on $X := \mathbb{R}^k$.

Definition A.24. The *Clarke generalized derivative* $\partial f(a)$ of f at $a \in X$ is the convex subset $\partial f(a)$ of $X^* = T_x^*X$ defined as follows: by Rademacher's theorem, the set $\text{dom}(df)$ of differentiability points of f is dense in X ; if $df := \{(x, df(x)) : x \in \text{dom}(df)\}$, we let

$$\partial f(a) := \text{co}\{y \in X^* : (a, y) \in \overline{df}\},$$

where co stands for the convex hull; in other words, $\partial f(a)$ is the convex hull of the set of limits of convergent sequences $df(x_n)$ with $\lim x_n = a$. As $|df(x)|$ is bounded by the local Lipschitz constant of f for x close to a , every sequence $df(x_n)$ with $\lim x_n = a$ is bounded and therefore has a convergent subsequence, implying

$$\forall a \in X \quad \partial f(a) \neq \emptyset;$$

moreover, $\partial f(a)$ is compact, being the convex hull of a compact subset. The subset

$$\partial f := \{(x, y) | y \in \partial f(x), x \in X\}$$

is a generalized version of the *enlarged pseudograph* defined for semi-concave functions in [Arn11], where the *pseudograph* is df . In simple one-dimensional cases, it is obtained by adding a vertical segment to df where f is not differentiable:

Remark A.25. The set $\partial f(x)$ consists of a single point if and only if f is “ C^1 at x with respect to the set where it is differentiable”.

Proposition A.26. *The set-valued function $x \mapsto \partial f(x)$ is upper semi-continuous: for every convergent sequence $(x_n, y_n) \rightarrow (x, y)$ with $y_n \in \partial f(x_n)$, one has $y \in \partial f(x)$. In other words, ∂f is closed.*

**The theory extends to reflexive Banach spaces [Cha81].

Definition A.27. A point $x \in X$ is called a *critical point* of f if $0 \in \partial f(x)$; the number $f(x)$ is then called a *critical value* of f . By Proposition A.26, the *critical set* $\text{Crit}(f)$ of f , consisting of its critical points, is closed in X .

Setting

$$\lambda(x) := \min_{w \in \partial f(x)} |w|_{X^*},$$

we say that f satisfies the *Palais-Smale condition* (P.S.) if every sequence (x_n) along which $f(x_n)$ is bounded, and such that $\lambda(x_n)$ goes to 0, possesses a convergent subsequence—whose limit is a critical point of f by Proposition A.26, as there is a sequence $y_n \in \partial f(x_n)$ converging to 0.

Example A.28. The P.S. condition is satisfied when $\text{Lip}(f - Q) < \infty$ for some *non-degenerate* quadratic form Q on X ; moreover, in that case, $\text{Crit}(f)$ is *compact*.

Proof. Indeed, if $\psi := f - Q$, each subset $\partial f(x) = \partial \psi(x) + dQ(x)$ consists of vectors whose norm is at least $|dQ(x)| - \text{Lip}(\psi)$, hence $\lambda(x) \geq |dQ(x)| - \text{Lip}(\psi)$, which tends to $+\infty$ when $|x| \rightarrow \infty$; therefore, there exists $R > 0$ such that every sequence (x_n) with $\lim \lambda(x_n) = 0$ satisfies $|x_n| \leq R$ for all large enough n , implying both the P.S. condition and the compactness of $\text{Crit}(f)$. \square

Theorem A.29 (Deformation Lemma I). *Suppose f satisfies the P.S. condition and let $f^c := \{x | f(x) \leq c\}$ for each $c \in \mathbb{R}$. If c is not a critical value of f , then there exist $\epsilon > 0$ and a bounded smooth vector field V on X equal to 0 off $f^{c+2\epsilon} \setminus f^{c-2\epsilon}$, and whose flow φ_V^t satisfies $\varphi_V^1(f^{c+\epsilon}) \subset f^{c-\epsilon}$.*

Theorem A.30 (Deformation Lemma II). *Suppose f satisfies the P.S. condition. If $c \in \mathbb{R}$ is a critical value of f and N any neighbourhood of $K_c := \text{Crit}(f) \cap f^{-1}(c)$, then there exist $\epsilon > 0$ and a bounded smooth vector field V on X equal to 0 off $f^{c+2\epsilon} \setminus f^{c-2\epsilon}$, whose flow φ_V^t satisfies $\varphi_V^1(f^{c+\epsilon} \setminus N) \subset f^{c-\epsilon}$.*

Lemma A.31 (Chain rule). *If $f : X \rightarrow \mathbb{R}$ is a Lipschitz function, $F : X \rightarrow X$ a C^1 diffeomorphism, then*

$$\partial(f \circ F)(x) = \partial f(F(x)) \circ dF(x) := \{dF(x)(\xi), \xi \in \partial f(F(x))\}$$

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